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Anomalies

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February 1999

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Abstract

I discuss the role of anomalies in the modern development of
quantum field theory and their implications for physics.

Introduction.

Symmetries play an essential role in our understanding of elementary particle physics. Global symmetries in the form of conserved charges label the physical states and reflect the existence of conserved local currents. Local symmetries in the form of gauge field theories are used to describe practically all aspects of elementary particle physics phenomena and imply the existence of vector gauge fields coupled to conserved local currents.

In electromagnetism, the photons are the quanta of the electromagnetic gauge field. In the theory of electroweak interactions, the massive W and Z particles are the quanta of the electroweak gauge fields in addition to the massless photon. The strong dynamics of the quarks and gluons are controlled by the color interactions of the quantum chromodynamic gauge fields. Local Lorentz symmetries are used to describe the gravitational interactions.

In some cases the symmetries are not realized explicitly although these invisible symmetries still involve exact symmetries at the fundamental level. In quantum chromodynamics, the color confinement phenomena results from an exact local color gauge symmetry. However color confinement implies that there are no asymptotic states with color, such as the fundamental quarks and gluons, and only color singlet particles can be directly observed as isolated states.

Symmetries can also be dynamically broken without destroying the exact underlying symmetry. Spontaneous magnetization occurs when the spins in a material tend to align in a particular direction breaking the explicit rotational symmetry. This spontaneous breaking of the rotational spin symmetry implies the existence of spin waves which govern the long range fluctuations of the spins. Chiral symmetries reflect the independent rotations of the left and right handed components of fermions which is an exact symmetry of a gauge field theory of massless Dirac fermions as in the case of quantum electrodynamics with massless electrons. PCAC and the dynamics of massless pions are thought to reflect the dynamical breaking of the approximate chiral symmetries of the strong interactions. At the fundamental level these global chiral symmetries are due to local gauge dynamics of the color interactions becoming exact in the limit where the light quarks are massless.

Local gauge symmetries can also be spontaneously broken. Superconductivity results from the dynamical breaking of the electromagnetic gauge symmetry. This dynamical breaking implies the existence of supercurrents and the Meissner effect which is related to the generation of a dynamical magnetic mass for the photon, the gauge quanta of the

1

electromagnetic gauge field.   In a similar manner, the electroweak interactions described by an exact local gauge symmetry which is dynamically broken generating masses for all of the presently observed particles including the massive gauge bosons, the W and Z particles, which mediate the observed electroweak forces.

At present the Standard Model is used to explain all of the observed phenomena of elementary particle physics.   The Standard Model is based on exact local gauge symmetries and the dynamics generated by the local gauge fields coupled to the corresponding local conserved currents.   The Standard Model currently invokes the local gauge symmetries,
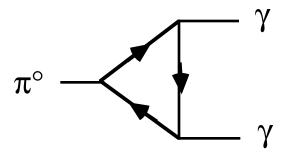
$$SU(3)_{color} \otimes (SU(2) \otimes U(1))_{ew} \otimes Gravity,$$

of the strong, electroweak and gravitation interactions.   Of course there are many speculations about additional dynamical structure including supersymmetry, technicolor grand unification and strings.

Anomalies - Clashing Symmetries.

Symmetries provide the fundamental framework for our present formulation of theoretical particle physics.   However, anomalies arise when apparent classical symmetries come in conflict.   This clashing of symmetries has an important impact on both the dynamics and the symmetry structure of the theories we use to describe elementary particles.   In the following sections, I will discuss the origins of anomalies, the structure of anomalies and some of the implications of anomalies for physics.

The original anomaly puzzle arose in attempts to apply the newly formulated quantum field theory to the two photon decay of the neutral pion [1],[2].   J. Steinberger computed the decay from the one loop, triangle diagram for a virtual proton with point couplings to the neutral pion and photons.   The decay amplitude seemed to depend strongly upon whether pseudoscalar or pseudovector couplings were used for the interaction of the pion with the proton.   However, these interactions seemed to be equivalent if one integrated by parts and used the field equations.   This contradiction between the naive application of the quantum field equations and the direct calculation of the triangle diagrams became known as the anomaly.   Although the pseudoscalar coupling eventually proved to give the correct experimental result, it is possible that this early attempt to apply quantum field theory to problems beyond QED convinced Steinberger to focus his future career on experimental physics instead of struggling with unreliable theories.

A formal resolution of the problem of the equivalence of pseudovector and pseudoscalar couplings was provided J. Schwinger [3] using proper time methods.   He showed that a careful definition of singular operator products was required before the equations of motion could be used to study the anomaly using the equations of motion of the quantum field theory.

A more complete understanding of the anomaly and its physical impact came from the study of the anomalous divergence equations for the axial-vector current [4],[5]. Axial-vector currents had become an important focus of research because of their role in understanding hadronic chiral symmetry or PCAC. The partial conservation of the axial-vector current followed from this chiral symmetry and implied particular couplings for the pions at low energy. Massless pions are identified as the Goldstone bosons of dynamical chiral symmetry breaking. Adler showed that the anomaly required the existence of specific operator corrections to the fermion axial-vector divergence equation.

$$\partial^{\mu}\{\overline{\psi}\gamma_{\mu}\gamma_5\psi\} = 2m\{\overline{\psi}i\gamma_5\psi\} + \frac{\alpha}{4\pi}F^{\mu\nu}\cdot{}^{*}F_{\mu\nu}$$

This result for a free fermion can be generalized to the axial-vector current for hadronic chiral symmetry. The anomaly modifies the divergence equation and predicts the decay amplitude for the Goldstone pion,

$$\partial^{\mu}J_{5\mu} = tr_{fermions}\{T^3Q^2\}\frac{\alpha}{4\pi}F^{\mu\nu}\cdot{}^{*}F_{\mu\nu}$$

$$A_{\pi^o\to\gamma\gamma} \to tr_{fermions}\{T^3Q^2\}/f_{\pi}$$

where the anomaly coefficient is determined by the fundamental fermion structure of the theory. The anomalous divergence equation implies that the axial vector current can not be conserved in the presence of electromagnetism even in the symmetric limit where the pions are massless. From this perspective, the chiral symmetry associated with axial-vector current clashes with local gauge symmetries of electromagnetism.
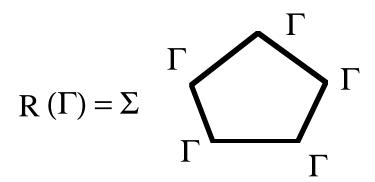
Because the magnitude of the pion decay amplitude is directly related to the strength of the anomaly, it is a sensitive measure of the fundamental fermion structure of a dynamical theory of hadrons. The measured values of the anomalous pion decay amplitude and the e+e- annihilation cross-section could be combined with current algebra and operator product expansion methods to provide the first convincing evidence for the dynamical color triplet quark picture [6]. Of course, the observed pion decay rate was also consistent with the original Steinberger calculation if pseudoscalar pion-nucleon couplings were used to compute the proton loop amplitude.

The Nonabelian Anomaly.

Anomalies have a more complex structure than the abelian anomaly observed in the anomalous divergence of the neutral axial-vector current. Nonabelian anomalies can be studied using generalized fermion loops for nonabelian currents where the fermions have arbitrary nonabelian couplings to vector, axial-vector, scalar and pseudoscalar densities.

$$L = \overline{\psi}\{\gamma^{\mu}V_{\mu} + \gamma^{\mu}\gamma_5 A_{\mu} - \Sigma - i\gamma_5\Pi\}\psi = \overline{\psi}\{\Gamma\}\psi$$

3

Explicit perturbative computations of general fermion loops for arbitrary external fields can be made where the short distance singularities are controlled by a well-defined cutoff or regularization procedure.

$$R\left(\Gamma\right) = \Sigma$$

This vacuum functional, or fermion loop effective potential, can be used to define consistent matrix elements of the nonabelian current and other operators. The covariant derivative of these currents can then be studied for possible anomalous terms. This study corresponds to an explicit check of the gauge covariance of the effective potential. Anomalous terms reflect the explicit breaking of the nonabelian gauge symmetries. A general regularization procedure will normally break many of these symmetries. Local counter-terms can then be added to the effective potential to restore the classical gauge symmetries. When this is not possible, the fermion loops are said to contain anomalies. By explicit calculation [7], all anomalous terms can be made to cancel except those involving certain external vector and axial-vector fields. For a particular choice of counter-terms, the gauge variation of the general fermion loop effective action be reduced to an especially simple form,

$$D(\Lambda_+,\Gamma) = R(\Gamma i\Lambda_+ - i\Lambda_-\Gamma - \gamma\cdot\partial\Lambda_+,\Gamma)$$

$$= \frac{1}{6}\frac{\pi^2}{(2\pi)^4}i\int dz \; \varepsilon_{\mu\nu\sigma\tau}tr\gamma_5\{2i\Lambda_+\partial^\mu V_+^\nu\partial^\sigma V_+^\tau - \partial^\mu V_+^\nu V_+^\sigma V_+^\tau\}$$

where Λ_+ is a left-handed gauge transformation and V_+ is the left-handed external gauge field. Right handed gauge transformations yield a corresponding result. The generalized anomalous divergence equation for nonabelian currents yields

$$D^\mu J_{+\mu}^a(x) = \frac{1}{6}\frac{\pi^2}{(2\pi)^4}\varepsilon_{\mu\nu\sigma\tau}tr\lambda_+^a\left\{2\partial^\mu V_+^\nu\partial^\sigma V_+^\tau - \partial^\mu(V_+^\nu V_+^\sigma V_+^\tau)\right\}$$

where D^μ is the appropriate covariant derivative. Since the anomalous divergence only involves other external gauge fields, the breaking of the nonabelian symmetries can be viewed as a clash between the symmetries associated with the current and the symmetries associated with the external gauge fields.

The form of the nonabelian anomaly is not arbitrary but is constrained by consistency conditions which must be satisfied by any proper formulation of the quantum theory [8]. The Wess-Zumino consistency conditions provide a powerful constraint on the algebraic

structure of the anomaly and a simple test for the consistency of any specific calculation of anomalous terms.

The general result for the fermion loop anomaly obtained above has been confirmed by many different methods. A particularly elegant derivation of the anomaly invokes the path integral formulation of quantum field theory [9]. Fermion loops are generated by the functional integral,

$$\int D\psi D\overline{\psi} \ \exp\left\{i\int dx\{\overline{\psi}i\gamma\cdot D\psi\}\right\}$$

where the classical fermion action is presumed to be covariant under generalized gauge transformations, but the fermionic measure may not preserve this covariance. Even here great care must be used in giving precise meaning to these formal expressions. In this formalism, anomalies are directly related to the noninvariance of the fermionic measure and not to problems associated with defining composite operators. Of course, this approach gives the same result as the direct calculation of the fermion loop diagrams, but it adds an important perspective to our understanding of anomalies.

Nonrenormalization Theorem.

A remarkable feature of anomalies concerns their behavior under renormalization. A careful study of higher order radiative corrections shows that these corrections do not modify the fermion loop anomaly computed above. Since anomalies reflect unavoidable gauge symmetry breaking, they are determined solely by the structure of the small fermion loops and their symmetries [10]. The nonrenormalization theorem was originally checked by explicit two loop computations and confirmed by general regularization arguments to all orders and extended to arbitrary renormalizable quantum field theories in four dimensions [10], [11]. The nonrenormalization theorem was also proven using renormalization group methods [12].

The nonrenormalization theorem is extremely important as it establishes the fundamental significance of the anomaly. The anomaly is not simply an artifact of a particular method of calculation or order of perturbation theory. As stated in our discussion of the evidence for color triplet quarks, the anomaly directly measures properties related to the fundamental fermion structure of the underlying quantum field theory. This feature has great significance in the many applications of anomalies to physics.

Classical Applications.

Anomalies have many different implications for quantum field theory. The consistency of gauge field theory requires the absence of anomalies associated with the dynamical currents which implies that the fermion loop anomalies must cancel between different kinds of fermions in the theory. If the anomalous current divergence involves dynamical gauge fields, then the global symmetries associated with the anomalous current are explicitly broken by dynamics of the gauge fields. Even if no dynamical currents are involved, anomalies can have important implications for the global current algebra associated with the external symmetries of a quantum field theory.

Anomaly Cancellation.   <DDD>

Anomalies reflect an intrinsic breaking of local gauge symmetries which can not be compensated by simply adding local counter-terms in higher order calculations.   Since gauge field theories are consistent only if the local gauge symmetries are preserved by the quantum theory, the presence of anomalies implies that certain gauge models simply do not exist at the quantum level.   Hence, anomalies associated with the dynamical gauge currents must cancel if the dynamical gauge symmetries are to be preserved.   The fermion loop anomalies depend only on the charge structure of the dynamical fermions, and their cancellation constrains the fermion matter content of many gauge field theories.   The nonrenormalization theorem then guarantees that this cancellation will be preserved to all orders.   From the form of the nonabelian anomaly, it can be shown that models with vectorlike gauge couplings, such as QED or QCD, do not have dynamical anomalies.   Only theories where the fermions have chiral gauge couplings can have nontrivial anomalies.

The Standard Model of the electroweak interactions provides an interesting example of a chiral gauge theory where anomalies do occur but are canceled between the various quark and lepton contributions [13], [14].   The anomalies for a single generation of quarks and leptons are listed below

Standard Model	Leptons	Quarks	Sum
$SU(2)^2 \otimes U(1)$	-1/2	3•(1/6)	0
$U(1)^3$	1-1/4	1/36-8/9+1/9	0

It is a remarkable feature of the Standard Model that a theory involving only quarks or only leptons would not be consistent, but the combined theory of quarks and leptons is free of all dynamical anomalies.   Anomaly cancellation is a central element in building models beyond the Standard Model including grand unification, extended technicolor or any other theory which adds new fermions or additional gauge interactions.

Global Symmetry Breaking. <GDD>

In gauge field theories, the anomalous divergence equations imply that various global symmetries can be broken by anomalies.   In the original calculation of the axial current anomaly, the chiral symmetry of the neutral pion current was broken by the coupling to the electromagnetic gauge fields which modified the low energy theorem for the coupling of pions to photons.

Global symmetries can be broken more dramatically by the presence of nontrivial gauge dynamics.   The U(1) problem of QCD is a classic example.   The original formulation of quark model seemed to have too much symmetry as there were nine conserved chiral currents in the limit where the light quarks are massless.   Weinberg had argued that there should be an extra Goldstone boson, an η', nearly degenerate with the pion. Instead, the physical η' has a mass of order 1 GeV.   In quantum chromodynamics, the singlet axial-vector current has an anomaly involving the QCD gauge fields. An explicit calculation by 't Hooft [15] showed that instanton effects could break the U(1) symmetries and generate a mass for the η' [16].

In a similar vein, instanton effects can be shown to generate explicit breaking of the baryon number symmetry in the Standard Model [15]. This may be somewhat surprising as the baryon number current is a vector current and not normally associated with anomalies. However, the Standard Model requires that the SU(2) X U(1) gauge symmetries be exactly preserved. Since these currents have chiral structure, the anomaly must be shifted away from the dynamical currents, and it reappears as an anomaly in the baryon number current. Hence, the anomaly predicts the proton will decay in the normal Standard Model although the explicit calculation shows that the vacuum decay rate is so highly suppressed that a proton has yet to decay via this mechanism in the entire lifetime of the universe.

Another implication of the QCD anomaly concerns the strong CP problem. Naively, all CP violating phases in the quark and lepton masses matrices can be rotated away leaving only the weak CP phases of the CKM matrix. However, the anomaly induced U(1) breaking of QCD implies that the U(1) phase cannot be freely rotated and a strong CP violation remains. Since there are precise limits on the size of any strong CP violation, alternative models beyond the standard model were considered where a new Peccei-Quinn symmetry [17] would allow the strong CP phase to be rotated away. However, Wilczek and Weinberg [18] argued that this new symmetry would imply the existence a new pseudo-Goldstone boson, the axion. Detailed predictions about the mass and couplings of the axion could be made using the anomalous current algebra reflecting the strong breaking of the U(1) symmetry in QCD [19]. Extensive tests of these predictions show that axions associated with the scale the electroweak interactions are now ruled out [20] and only much higher scales are consistent with the axion picture. The resolution of the strong CP problem remains an outstanding puzzle of the Standard Model.

Global Current Algebra. <GGG>

Anomalies also modify the current algebra relations associated with purely global symmetries. This is clear from the anomalous divergence equation where the external gauge fields in the anomalous divergence are associated with global symmetry currents and not the dynamical gauge fields. These anomalies reflect the clash of symmetries generated by the quantum effects of the fermion loops. In many applications of current algebra one combines the constraints of current algebra with low energy theorems associated with the infrared dynamics of the system. Wess and Zumino used their consistency conditions to derive an effective action for the Goldstone pions consistent with the anomalous couplings to the electromagnetic field [8]. Witten showed that this could be extended to derive anomalous terms in the purely strong strong dynamics of pseudoscalar mesons [21].

The anomaly has both ultraviolet and infrared implications. The anomaly associated with the global symmetries of a given theory provides a set of consistency conditions which must be satisfied by any infrared realization of the theory. These consistency conditions place severe constraints on the massless spectrum of fermions and Goldstone bosons even when the dynamics is highly nonperturbative [22].

Topology and Geometry.

Anomalies have important relations to the topology and geometry of gauge fields. Atiyah and Singer [23] showed that index theorems and the spectral properties of the Dirac operator relate the anomaly to the topological structure of gauge fields. The eigenvalues of the Dirac operator,

$$\gamma \bullet D = \gamma^\mu D_\mu = \gamma^\mu \partial_\mu - i T^a \gamma^\mu A_\mu^a$$

depend upon the deformations of the background gauge fields and reflect their topological structure. The anomalous divergence of the axial vector current,

$$\partial^\mu J_{5\mu} = \frac{N_f}{8\pi^2} tr\left\{ G^{\mu\nu}(A) \cdot {}^* G_{\mu\nu}(A) \right\}$$

is directly related to the topological index of the gauge field

$$\nu = \frac{1}{16\pi^2} \int dz \ tr\left\{ G^{\mu\nu}(A(z)) \cdot {}^* G_{\mu\nu}(A(z)) \right\}$$

which takes on integer values.

Differential geometry has been used to analyze the structure of anomalies in arbitrary dimensions of space-time [24]. The descent equations can be used to connect various aspects of the anomaly structure. As in the case of the Wess-Zumino consistency conditions, the descent equations strongly constrain the anomalous structure allowed for any theory.

The anomaly also implications for topological objects which occur in gauge field theories. Instantons, sphalerons and similar objects are related to fermion number changing processes which are determined by the anomaly structure of the underlying theory [22]. Anomalies are related to the mechanisms of charge fractionalization and induced charge on topological defects such as dyons, skyrmeons and polyacetylene. Anomalies also have an important impact on the physics of magnetic monopoles, cosmic strings, domain walls, vacuum bubbles, D-branes, etc. In many cases where the physics is highly nonperturbative, the anomaly structure provides the only precise information on the behavior of complex systems.

Gravitational Anomalies.

Anomalies also occur for systems interacting with gravitational fields. In precise analogy with the axial-vector current anomaly in a background electromagnetic field, the fermion loop processes generate a gravitational anomaly in the divergence of the axial-vector current [26],

$$\partial^\mu J_{5\mu} = \frac{1}{768\pi^2} \varepsilon_{\mu\nu\sigma\tau} \ R^{\mu\nu\alpha\beta} R_{\alpha\beta}^{\sigma\tau}$$

where the anomaly is related to a topological index of the gravitational field. Since the Standard Model contains chiral U(1) currents, the potential for gravitational anomalies exists. Such an anomaly would imply a clash between the Standard Model gauge symmetries and the general covariance of the background gravitational field. We would expect the gravitational anomalies to cancel if we wish to preserve our normal picture of gravity. In the Standard model, the individual fermions do have anomalous contributions,

Standard Model	Leptons	Quarks	Sum
$R^2 \otimes U(1)$	$2(-\frac{1}{2})+1$	$3(\frac{1}{3})+3(-\frac{2}{3})+6(\frac{1}{6})$	0

but the sum over all fermionic contributions cancels. Contrary to the case of the gauge anomalies, the cancellation occurs separately for quarks and leptons.

Pure gravitational anomalies can also exist in 2, 6 and 10 dimensions [26]. As in gauge theories, it is important to determine the precise form of the consistent gravitational anomaly as distinguished from the covariant anomalies associated with various currents or densities. In theories with fermions, the vierbein field must be introduced to define the spin using the tangent space symmetries. In this case, the local gravitational symmetries can be viewed from the perspectives of local Lorentz symmetry or general covariance. By using the veirbein field, the gravitational anomalies can be transformed from one perspective to the other by adding the analogue of Wess-Zumino counter-terms to the gravitational action [27].

Supersymmetry.

Supersymmetry adds additional complexity to the anomaly picture. Here there is potential for the gauge symmetries or global symmetries to clash with supersymmetry. Indeed, there was initially considerable confusion between the nonrenormalization theorem associated with the axial-vector currents, the renormalization of the supersymmetric β-function and the nonrenormalization theorems associated with the holomorphy of the superpotential [27], [28], [29],[30]. Anomalies also have an important impact on the nonperturbative structure of the superpotential, holomorphy and duality [31].

Superstrings.

The modern superstring era began in 1984 with the observation by Green and Schwarz [32] that the anomalies which affected earlier formulations of string theory could be made to cancel. The apparent loop anomalies were found to cancel against anomalous couplings of the graviational sector. Consistent superstring theories were found to exist in 10 dimensions (four visible dimensions and six compact dimensions) for particular gauge groups. The most interesting early string model was the heterotic string [33]. The low energy spectrum of the theory is determined by anomalies in terms of index theorems and the topological structure of the compact six dimensional manifolds. In this way the anomalies could be used to predict the generation structure of the chiral fermions [34]. More recently, theoretical efforts have focused on superstring duality, M-theory and D-branes [34]. Even here anomalies and related phenomena continue provide important insights into the structure and applications of string theory.

Conclusions.

Anomalies started out as a troublesome ambiguity about how to apply the new ideas of quantum field theory to interesting physical problems. The resolution of this ambiguity led to a more fundamental understanding of quantum field theories and their symmetries. The discovery and analysis of the complete nonabelian anomaly showed that the anomaly was much more complex than the simple form of the anomalous divergence of the axial-vector current. Anomalies could be viewed as the fundamental clash between the classical symmetries which can occur in a quantum system. The nonrenormalization theorems showed that the anomalies reflected the fundamental structure of the quantum field theory and were not just an artifact of a particular computation in some order in perturbation theory. As nonabelian gauge theories began to take over the theoretical foundations of particle physics, the anomaly played an important role in determining the structure of the gauge models and the symmetry structure of the resulting theories. Anomalies cancellation was a required condition for model building, the global symmetry structure is modified by the presence of anomalies, and the anomaly also changed the global current algebras. In many cases, the anomaly provides the only nonperturbative information about specific gauge field theories, as reflected by the constraints of the 't Hooft anomaly matching conditions and by many other applications.

Connections to fundamental mathematical structures have led to a deeper understanding of anomalies and their implications Differential geometry provided an elegant mechanism for the analysis of anomaly structure and pointed to generalizations of the anomaly picture. Index theorems, spectral flow and related techniques revealed the deep connection between anomalies and the topological structure of gauge fields. The interplay between the mathematics and the physics has led to a much richer view of both fields.

Anomalies played an important role in the rebirth of string theory. They continue to have an important impact on recent developments of string theory, M-theory and D-branes. String theories have revealed a much richer symmetry structure that goes far beyond the symmetries of normal gauge field theory, and anomalies may help provide a path to a more complete understanding of the symmetries and the dynamics.

In this talk I have only provided a very limited view of anomalies and their applications. Many people have played important roles in understanding the mathematical structure of anomalies and in developing the vast array of applications in both physics and mathematics. In my original derivation of the nonabelian anomaly, I knew the result had fundamental significance but had little idea how pervasive anomalies would become in the future.

Acknowledgments.

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